pressure equation developed in Ref. 7 would seem to be of most significance to the performance analyst in more carefully approximating the ground overpressure experienced during the vehicle acceleration-climb.

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Out-of-Plane Perturbations of a Circular Satellite Orbit

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The method of differential coefficients as applied to trajectory prediction and guidance analysis for space missions hinges on the validity of the approximations of first-order error analysis.[1, 2] In applying the linearity assumption to the variation of parameters for space trajectories, the in-plane and out-of-plane parameters are conveniently separated because small variations in in-plane parameters produce no first-order effect on the out-of-plane parameters, and vice versa.

It is the purpose of this note to investigate the validity of the first-order approximation concerning the out-of-plane motion of a long-lifetime satellite. Using a circular orbit for simplicity, it is shown that the effect of small deviations in in-plane parameters on the statistics of the out-of-plane motion may become significant, indicating the need of considering higher-order terms in certain cases.

Based on the linear perturbation theory, the perturbations in position and velocity perpendicular to the standard orbital plane at any time t are given by

$$\left.\begin{aligned} \delta z(t) &= \frac{\partial z(t)}{\partial z_0}\,\delta z_0 + \frac{\partial z(t)}{\partial \dot z_0}\,\delta \dot z_0 \\ \delta \dot z(t) &= \frac{\partial \dot z(t)}{\partial z_0}\,\delta z_0 + \frac{\partial \dot z(t)}{\partial \dot z_0}\,\delta \dot z_0 \end{aligned}\right\} \qquad (1)$$

where δz_0 and $\delta \dot z_0$ are small variations in the out-of-plane position and velocity at time t_0, respectively. The partial derivatives are evaluated along the standard trajectory. The quantities δz_0 and $\delta \dot z_0$ can be thought of, for instance, as random errors due to guidance error sources at the end of the powered-flight phase (or injection).

For the case of circular orbit, Eqs. (1) take the forms

$$\left.\begin{aligned} \delta z(t) &= \cos n(t-t_0)\delta z_0 + \frac{\sin n(t-t_0)}{n}\,\delta \dot z_0 \\ \delta \dot z(t) &= -n\sin n(t-t_0)\delta z_0 + \cos n(t-t_0)\delta \dot z_0 \end{aligned}\right\} \qquad (2)$$

where $n = v/r = \text{const}$, the r and v being, respectively, the radial distance and speed on the standard circular orbit. Assuming that δz_0 and $\delta \dot z_0$ are statistically independent with zero means and variances σ_{z0}^2 and $\sigma_{\dot z0}^2$, the variances and covariance of $\delta z(t)$ and $\delta \dot z(t)$, the quantities of primary interest, are easily determined from the equations

$$\left.\begin{aligned} \sigma_z^2(t) &= \sigma_{z0}^2\cos^2 n(t-t_0) + \frac{\sigma_{\dot z0}^2 \sin^2 n(t-t_0)}{n^2} \\ \sigma_{\dot z}^2(t) &= \sigma_{z0}^2\sin^2 n(t-t_0) + \sigma_{\dot z0}^2\cos^2 n(t-t_0) \\ \mu_{z\dot z}(t) &= \frac{1}{2}\left(-n\sigma_{z0}^2 + \frac{\sigma_{\dot z0}^2}{n}\right)\sin 2n(t-t_0) \end{aligned}\right\} \qquad (3)$$

Although it is clear that pure variations in in-plane parameters produce no perturbations in the out-of-plane direction, it is of interest to consider their influence on the behavior of out-of-plane motion in the presence of small variations δz_0 and $\delta \dot z_0$. Hence, let us reconsider Eqs. (2), taking into account that the in-plane parameters r and v are also randomly distributed within a small range about their respective standard values due to, for instance, guidance errors at injection.

For the purpose of comparison, consider r and v as random variables with the assumption that they are correlated in such a way that circularity of the orbit is conserved, i.e., $N = v/r$ is a random constant (independent of time). Although this assumption limits the consideration of a very special class of orbits, it permits examination of the problem effectively in a sample fashion.

In the numerical calculation that follows, assume that $\delta z_0 = 0$ with probability one (deterministic), that $\delta \dot z_0$ has mean zero and variances $\sigma_{\dot z0}^2$, and that the distribution of N is a discrete one approximating a one-dimensional normal distribution. Write

$$N = n(1 + X) \qquad (4)$$

where X is a dimensionless random variable with mean zero, and the probability of taking the values $\pm 3\epsilon$, $\pm 2\epsilon$, $\pm\epsilon$, and 0 are, respectively, 0.0060, 0.0606, 0.2417, and 0.3834. The standard deviation of X is 0.052, with $\epsilon = 0.05$. Further assume that X and $\delta \dot z_0$ are statistically independent.

Equations (3) now have the forms, according to linear perturbation approximation,

$$[n^2/\sigma_{\dot z0}^2]\sigma_z^2(t) = \sin^2 n(t-t_0) \qquad (5a)$$

$$[1/\sigma_{\dot z0}^2]\sigma_{\dot z}^2(t) = \cos^2 n(t-t_0) \qquad (5b)$$

$$[n/\sigma_{\dot z0}^2]\mu_{z\dot z}(t) = \tfrac{1}{2}\sin 2n(t-t_0) \qquad (5c)$$

By including in-plane variations, they become

$$\left[\frac{n^2}{\sigma_{\dot z0}^2}\right]\sigma_z^2(t) = \sum_j \left\{\frac{p_j \sin^2[(1+X_j)n(t-t_0)]}{(1+X_j)^2}\right\} \qquad (6a)$$

$$\left[\frac{1}{\sigma_{\dot z0}^2}\right]\sigma_{\dot z}^2(t) = \sum_j p_j \cos^2[(1+X_j)n(t-t_0)] \qquad (6b)$$

$$\left[\frac{n}{\sigma_{\dot z0}^2}\right]\mu_{z\dot z}(t) = \sum_j \left\{\frac{p_j \sin[2(\ + X_j)n(t-t_0)]}{2(1+X_j)}\right\} \qquad (6c)$$

where p_j is the probability of X taking the value X_j.

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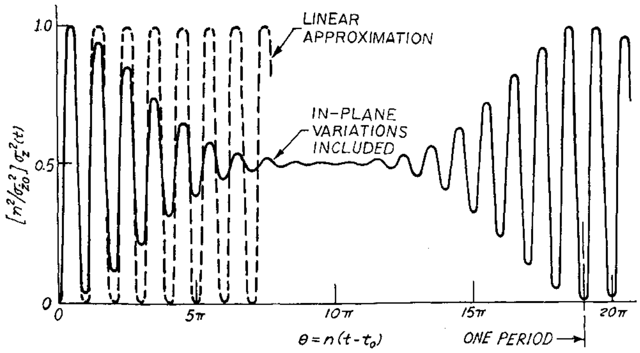

Fig. 1 The variance of out-of-plane position.

Equations (5a) and (6a) are compared graphically in Fig. 1, where the variance $\sigma_z^2(t)$ is plotted as a function of $\theta = n(t - t_0)$. We can readily observe a significant change of the time behavior of the variance due to small dispersions in in-plane parameters. Instead of regular oscillatory property as predicted by linear approximation, the behavior of the variance $\sigma_z^2(t)$ is that analogous to the "beat" phenomenon commonly observed in the vibration of engineering systems. The beat period is approximately $19\pi/n$ for the statistical model assumed for N.

It is clear from Eqs. (6b) and (6c) that the observations just discussed described equally well the behaviors of $\sigma_{\dot{z}}^2(t)$ and $\mu_{z\dot{z}}(t)$.

The restricted problem considered in this note serves to demonstrate that the statistical behavior of the out-of-plane motion of a satellite can deviate significantly from that predicted by linear perturbation theory. For satellites of long lifetimes in particular, errors in in-plane parameters may become important and cannot be ignored in predicting the out-of-plane perturbations.

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Effect of Temperature on Pressure Measurements

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Nomenclature

Kn = Knudsen number = λ/r
p = pressure
r = inside radius of the tube
T = absolute temperature
λ = mean free path of the gas

Subscripts

c = cold end of the tube
h = hot end of the tube
l = large tube
s = small tube

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Knudsen[1] has shown that, when two containers at unequal temperatures are connected by a tube, the pressures in the two containers do not become equal if the mean free path of the gas λ is such that $\lambda/r > \frac{1}{100}$ (approximately), where r is the radius of the tube. He defined the equilibrium condition in the tube as being one of zero net mass transfer across a section rather than an equality of pressure. The condition of zero net mass transfer can exist when a flow from the cold to hot end along the wall of the tube is exactly balanced by a flow from hot to cold along the axis of the tube. For small values of mean free path the central flow predominates, and the condition of pressure equality may be attained. When the mean free path is large, the flow close to the wall exercises a significant effect. In the limiting case where the mean free path is very much larger than the tube diameter, the pressures may be related by the approximate, free-molecule flow equation:

$$p_c/p_h = (T_c/T_h)^{1/2} \tag{1}$$

Knudsen conducted experiments to determine the pressure and temperature relationship in the continuum and transition regimes. As a result of these tests, he derived the following relationship:

$$dp/dT = (p/2T)\{1 + 2.46(Kn + 3.15)/[Kn(Kn + 24.6)]\}^{-2} \tag{2}$$

This equation was derived from experiments with hydrogen in glass tubes where the temperature difference between the ends of the tube was small. Howard[2] extended this work to stainless-steel tubing containing air with a much larger temperature difference between the two ends of the tube. For Knudsen numbers greater than two, the agreement between Howard's data and Knudsen's semiempirical equation is not good.

Because of the uncertainty as to the form of the pressure-temperature relationship for Knudsen numbers greater than two, an experimental program designed to better determine this variation has been carried out by the authors. A complete description of the experimental apparatus and procedure is contained in two test reports.[3,4] Basically, the apparatus consists of two tubes, one large and the other small, joined together as shown in Fig. 1. The temperature of this junction can be controlled by an electrical resistance heater. The other ends of the two tubes were water cooled. Pressures at the cold ends of the tubes were measured with a well-calibrated, low-range, pressure transducer. The temperatures at the hot and cold ends of the tubes were measured with chromel-alumel and copper-constantan thermocouples, respectively.

Inspection of Knudsen's semiempirical equation, Eq. (2), indicates that, for a fixed temperature ratio T_h/T_c, the pressure ratio p_c/p_h is a function of Knudsen number only. This indicates a method of analyzing the test data. Con-

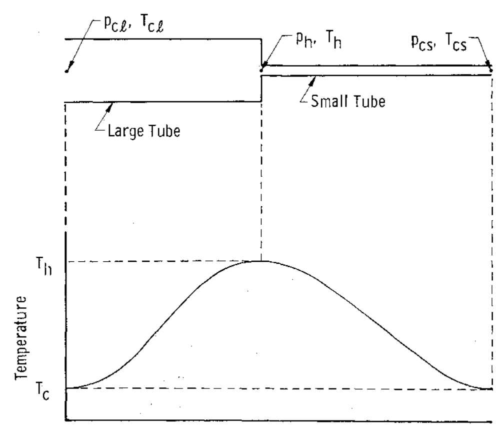

Fig. 1 Basic plan for the experiment.